Lecture Notes for PHY 405
Classical Mechanics

From Thorton & Marion's *Classical Mechanics*

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Chapter 11: Dynamics of Rigid Bodies

rigid body = a collection of particles whose relative positions are fixed.

We will ignore the microscopic thermal vibrations occurring among a 'rigid' body's atoms.

The inertia tensor

Consider a rigid body that is composed of N particles.
Give each particle an index $\alpha = 1 \ldots N$.
Particle α has a velocity $\mathbf{v}_{f,\alpha}$ measured in the *fixed* reference frame
and velocity $\mathbf{v}_{r,\alpha} = d\mathbf{r}_\alpha/dt$ in the reference frame that rotates about axis $\vec{\omega}$.
Then according to Eq. 10.17:

$$\mathbf{v}_{f,\alpha} \;=\; \mathbf{V} + \mathbf{v}_{r,\alpha} + \vec{\omega} \times \mathbf{r}_\alpha$$

where $\mathbf{V}$ = velocity of the moving origin relative to the fixed origin.

Note that the particles in a rigid body have zero relative velocity, ie, $\mathbf{v}_{r,\alpha} = 0$.

$$\begin{aligned}
\text{Thus} \quad \mathbf{v}_\alpha &= \mathbf{V} + \vec{\omega} \times \mathbf{r}_\alpha \quad \text{after dropping the f subscript} \\
\text{and} \quad T_\alpha &= \frac{1}{2} m_\alpha \mathbf{v}_\alpha^2 = \frac{1}{2} m_\alpha (\mathbf{V} + \vec{\omega} \times \mathbf{r}_\alpha)^2 \quad \text{is the particle's KE} \\
\text{so} \quad T &= \sum_{\alpha=1}^{N} T_\alpha \quad \text{the system's total KE is} \\
&= \frac{1}{2} M \mathbf{V}^2 + \sum_\alpha m_\alpha \mathbf{V} \cdot (\vec{\omega} \times \mathbf{r}_\alpha) + \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \mathbf{r}_\alpha)^2 \\
&= \frac{1}{2} M \mathbf{V}^2 + \mathbf{V} \cdot \left(\vec{\omega} \times \sum_\alpha m_\alpha \mathbf{r}_\alpha \right) + \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \mathbf{r}_\alpha)^2
\end{aligned}$$

where $M = \sum_\alpha m_\alpha =$ the system's total mass.

Now choose the system's center of mass $\mathbf{R}$ as the origin of the moving frame.

Since $\mathbf{r}'_\alpha = \mathbf{R} + \mathbf{r}_\alpha$ is α's position in the fixed reference frame,
$\Rightarrow \mathbf{R} = \sum_\alpha m_\alpha \mathbf{r}'_\alpha / M = \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}_\alpha)/M = \mathbf{R} + \sum_\alpha m_\alpha \mathbf{r}_\alpha / M$.
$\Rightarrow \sum_\alpha m_\alpha \mathbf{r}_\alpha = 0$ and the middle term in T is zero:

$$\begin{aligned}
\text{thus} \quad T &= T_{trans} + T_{rot} \\
\text{where} \quad T_{trans} &= \frac{1}{2} M \mathbf{V}^2 = \text{KE due to system's } \textit{translation} \\
\text{and} \quad T_{rot} &= \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \mathbf{r}_\alpha)^2 = \text{KE due to system's } \textit{rotation}
\end{aligned}$$

Now focus on T_{rot},
and note that $(\mathbf{A} \times \mathbf{B})^2 = A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2$ $\leftarrow$ see page 28 for proof.

Thus

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha[\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\mathbf{r}_\alpha)^2]$$

In Cartesian coordinates,

$$\begin{aligned}
\vec{\omega} &= \omega_x\hat{\mathbf{x}} + \omega_y\hat{\mathbf{y}} + \omega_z\hat{\mathbf{z}} \\
\text{so} \quad \omega^2 &= \omega_x^2 + \omega_y^2 + \omega_z^2 \equiv \sum_{i=1}^{3}\omega_i^2 \\
\text{and} \quad \mathbf{r}_\alpha &= \sum_{i=1}^{3} x_{\alpha,i}\hat{\mathbf{x}}_i \\
\text{so} \quad \vec{\omega}\cdot\mathbf{r}_\alpha &= \sum_{i=1}^{3}\omega_i x_{\alpha,i}
\end{aligned}$$

Thus

$$T_{rot} = \frac{1}{2}\sum_{\alpha} m_\alpha\left[\left(\sum_{i=1}^{3}\omega_i^2\right)\left(\sum_{k=1}^{3}x_{\alpha,k}^2\right) - \left(\sum_{i=1}^{3}\omega_i x_{\alpha,i}\right)\left(\sum_{j=1}^{3}\omega_j x_{\alpha,j}\right)\right]$$

We can also write

$$\begin{aligned}
\omega_i &= \sum_{j=1}^{3}\omega_j\delta_{i,j} \\
\text{where} \quad \delta_{i,j} &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\
\text{so} \quad \omega_i^2 &= \omega_i\sum_{j=1}^{3}\omega_j\delta_{i,j} = \sum_{j=1}^{3}\omega_i\omega_j\delta_{i,j} \\
\text{and also note that} \left(\sum_i a_i\right)\left(\sum_i b_i\right) &= \sum_i\sum_j a_i b_i \equiv \sum_{i,j} a_i b_i
\end{aligned}$$

$$
\begin{aligned}
\text{so} \quad T_{rot} &= \frac{1}{2}\sum_\alpha m_\alpha \left[\sum_{i,j}\omega_i\omega_j\delta_{i,j}\left(\sum_{k=1}^{3}x_{\alpha,k}^2\right) - \sum_{i,j}\omega_i\omega_j x_{\alpha,i}x_{\alpha,j}\right] \\
&= \frac{1}{2}\sum_\alpha m_\alpha \sum_{i,j}\omega_i\omega_j\left[\delta_{i,j}\sum_{k=1}^{3}x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right] \\
&= \frac{1}{2}\sum_{i,j}\omega_i\omega_j \sum_\alpha m_\alpha\left[\delta_{i,j}\sum_{k=1}^{3}x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right] \\
&\equiv \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} I_{i,j}\omega_i\omega_j \\
\text{where} \quad I_{i,j} &\equiv \sum_\alpha m_\alpha\left[\delta_{i,j}\sum_{k=1}^{3}x_{\alpha,k}^2 - x_{\alpha,i}x_{\alpha,j}\right]
\end{aligned}
$$

are the 9 elements of a 3×3 matrix called the *inertial tensor* $\{\mathbf{I}\}$.

Note that the $I_{i,j}$ has units of mass$\times$length^2.

Since $x_{\alpha,1} = x_\alpha$, $x_{\alpha,2} = y_\alpha$, $x_{\alpha,3} = z_\alpha$,

$$
\{\mathbf{I}\} = \left\{ \begin{array}{ccc}
\sum_\alpha m_\alpha(y_\alpha^2 + z_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\
-\sum_\alpha m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha(x_\alpha^2 + z_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\
-\sum_\alpha m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2)
\end{array} \right\}
$$

Be sure to write your matrix elements $I_{i,j}$ consistently;
here, the i index increments down along a column
while the j index increments across a row.

Also note that $I_{i,j} = I_{j,i} \Rightarrow$ the inertia tensor is symmetric.

The diagonal elements $I_{i,i}$ = the *moments* of inertia
while the off–diagonal elements $I_{i\neq j}$ =the *products* of inertia

For a continuous body having a density $\rho(\mathbf{r})$,

$$\begin{aligned} dI_{i,j} &\rightarrow \left(\delta_{i,j}\sum_{k=1}^{3} x_k^2 - x_i x_j\right) dm \\ &= \text{the contribution to } I_{i,j} \text{ due to mass element } dm = \rho dV \\ \text{thus} \quad I_{i,j} &= \int_V dI_{i,j} = \int_V \rho(\mathbf{r}) \left(\delta_{i,j}\sum_{k=1}^{3} x_k^2 - x_i x_j\right) dV \end{aligned}$$

where the integration proceeds over the system's volume V.

Example 11.3

Calculate $\{\mathbf{I}\}$ for a cube of mass M and width b and uniform density $\rho = M/b^3$. Place the origin at one corner of the cube.

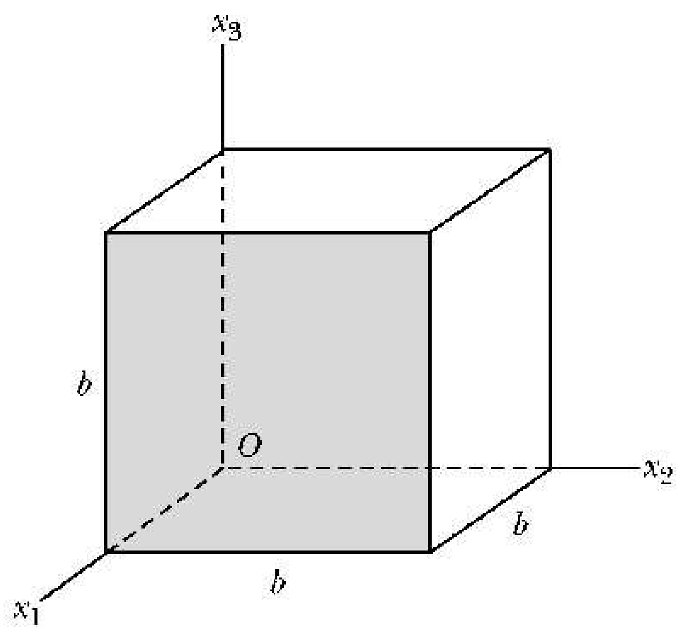

Fig. 11-3.

$$
\begin{aligned}
I_{1,1} &= \rho \int_0^b dx \int_0^b dy \int_0^b dz (x^2 + y^2 + z^2 - x^2) \\
&= \rho b \int_0^b dy (y^2 b + \frac{1}{3} b^3) \\
&= \rho b (\frac{1}{3} b^4 + \frac{1}{3} b^4) \\
&= \frac{2}{3} \rho b^5 = \frac{2}{3} M b^2
\end{aligned}
$$

It is straightforward to show that the other diagonal elements are $I_{2,2} = I_{3,3} = I_{1,1}$.

The off–diagonal elements are similarly identical:

$$
\begin{aligned}
I_{1,2} &= -\rho \int_0^b dx \int_0^b dy \int_0^b dz \cdot xy \\
&= -\rho \left(\frac{1}{2} b^2 \right) \left(\frac{1}{2} b^2 \right) b = -\frac{1}{4} \rho b^5 = -\frac{1}{4} M b^2 \\
&= I_{i \neq j}
\end{aligned}
$$

Thus this cube has an inertia tensor

$$
\{\mathbf{I}\} = M b^2 \left\{ \begin{array}{rrr} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{array} \right\}
$$

Principal Axes of Inertia

The axes $\hat{\mathbf{x}}_i$ that diagonalizes a body's inertia tensor $\{\mathbf{I}\}$ is called the *principal axes of inertia.*

In this case

$$
\begin{aligned}
I_{i,j} &= I_i \delta_{i,j} \\
\text{so} \quad T_{rot} &= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} I_{i,j}\omega_i\omega_j = \text{ rotational KE} \\
&= \frac{1}{2}\sum_{i=1}^{3} I_i \omega_i^2
\end{aligned}
$$

where the three I_i are the body's *principal moments of inertia.*

If this body is also rotating about one of its principal axis, say, the $\hat{\mathbf{z}}$, then $\vec{\omega} = \omega\hat{\mathbf{z}}$ and

$$T_{rot} = \frac{1}{2} I_z \omega^2$$

which is the familiar formula for the rotational KE in terms of inertia I_z found in the elementary textbooks.

$\Rightarrow$those formulae with $T_{rot} = \frac{1}{2} I\omega^2$ are only valid when the rotation axis $\vec{\omega}$ lies along one of the body's three principal axes.

The procedure for determining the orientation of a body's principal axes of inertia is given in Section 11.5.

Angular Momentum

The system's total angular momentum relative to the moving/rotating origin $\mathcal{O}_{rot}$ is

$$\mathbf{L} \;=\; \sum_{\alpha} m_{\alpha}\mathbf{r}_{\alpha} \times \mathbf{v}_{\alpha}$$

where $\mathbf{r}_{\alpha}$ and $\mathbf{v}_{\alpha}$ are α'a position and velocity relative to $\mathcal{O}_{rot}$.

Recall that $\mathbf{v}_{\alpha} = \vec{\omega} \times \mathbf{r}_{\alpha}$ (Eqn. 10.17), so

$$\mathbf{L} \;=\; \sum_{\alpha} m_{\alpha}\mathbf{r}_{\alpha} \times (\vec{\omega} \times \mathbf{r}_{\alpha})$$

Chapter 1 and problem 1-22 show that

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) \;&=\; A^2\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{A} \\ \text{so} \quad \mathbf{L} \;&=\; \sum_{\alpha} m_{\alpha}[r_{\alpha}^2\vec{\omega} - (\mathbf{r}_{\alpha} \cdot \vec{\omega})\mathbf{r}_{\alpha}]\end{aligned}$$

and the i^{th} component of $\mathbf{L}$ is

$$L_i \;=\; \sum_{\alpha} m_{\alpha}[r_{\alpha}^2\omega_i - (\mathbf{r}_{\alpha} \cdot \vec{\omega})x_{\alpha,i}]$$

And again use

$$r_{\alpha}^2 = \sum_{k=1}^{3} x_{\alpha,k}^2 \qquad \mathbf{r}_{\alpha} \cdot \vec{\omega} = \sum_{j=1}^{3} x_{\alpha,j}\omega_j \qquad \omega_i = \sum_{j=1}^{3} \omega_j\delta_{i,j}$$

So

$$
\begin{aligned}
L_i &= \sum_\alpha m_\alpha \left(\sum_{k=1}^{3} x_{\alpha,k}^2 \sum_{j=1}^{3} \omega_j \delta_{i,j} - \sum_{j=1}^{3} x_{\alpha,j} \omega_j x_{\alpha,i} \right) \\
&= \sum_{j=1}^{3} \omega_j \sum_\alpha m_\alpha \left(\delta_{i,j} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \\
&= \sum_{j=1}^{3} I_{ij} \omega_j
\end{aligned}
$$

where the $I_{i,j}$ are the elements of the inertial tensor $\{\mathbf{I}\}$.

Note that

$$
\begin{aligned}
\mathbf{L} &= \sum_{i=1}^{3} L_i \hat{\mathbf{x}}_i = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij} \omega_j \hat{\mathbf{x}}_i \\
&= \{\mathbf{I}\} \cdot \vec{\omega}
\end{aligned}
$$

Lets confirm this last step by explicitly doing the matrix multiplication:

$$
\begin{aligned}
\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&= \begin{pmatrix} I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{pmatrix} = \begin{pmatrix} \sum_j I_{1j}\omega_j \\ \sum_j I_{2j}\omega_j \\ \sum_j I_{3j}\omega_j \end{pmatrix}
\end{aligned}
$$

which confirms $\mathbf{L} = \{\mathbf{I}\} \cdot \vec{\omega}$.

Evidently the inertia tensor related the system's angular momentum $\mathbf{L}$ to its rotation $\vec{\omega}$.

Now examine

$$\begin{aligned}\sum_{i=1}^{3} \frac{1}{2}\omega_i L_i &= \frac{1}{2}\vec{\omega} \cdot \mathbf{L} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\omega_i\omega_j \\ &= T_{rot} = \text{ the body's KE due to its rotation} \\ \Rightarrow T_{rot} &= \frac{1}{2}\vec{\omega} \cdot \mathbf{L} = \frac{1}{2}\vec{\omega} \cdot \{\mathbf{I}\} \cdot \vec{\omega}\end{aligned}$$

Note that the RHS is a scalar (as is should) since
$\{\mathbf{I}\} \cdot \vec{\omega}$ is a vector, and a vector·vector = scalar.

Chapter 9 showed that if an external torque $\mathbf{N}$ is applied to the body, then

$$\mathbf{N} = \frac{d\mathbf{L}}{dt} = \{\mathbf{I}\} \cdot \dot{\vec{\omega}}$$

since $\{\mathbf{I}\}$ is a constant.

We now have several methods available for obtaining the equation of motion:
Newton's Laws,
Lagrange equations,
Hamilton's equation,
and $\mathbf{N} = \{\mathbf{I}\} \cdot \dot{\vec{\omega}}$ (which follows from Newton's Laws)

the latter equation is usually quite handy for rotating rigid bodies.

Example 11.4

A plane pendulum with two masses: m_1 lies at one end of the rod of length b, while m_2 is at the midpoint. What is the frequency of small oscillations?

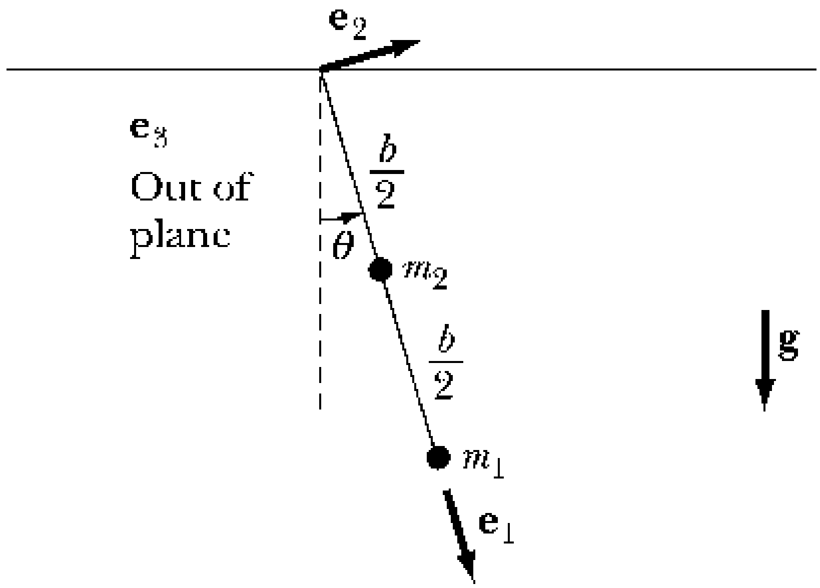

Fig. 11-5.

This solution will use $\mathbf{N} = \{\mathbf{I}\} \cdot \dot{\vec{\omega}}$.
Note that $\vec{\omega} = \dot{\theta}\hat{\mathbf{z}}$ which point out of the page,
so $\dot{\vec{\omega}} = \ddot{\theta}\hat{\mathbf{z}}$.

The inertia tensor has elements

$$
\begin{aligned}
I_{ij} &= \sum_\alpha m_\alpha \left(\delta_{i,j} \sum_{k=1}^{3} x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right) \\
\text{so} \quad I_{11} &= 0 \\
I_{22} &= m_1(b^2 - 0) + m_2 \left[\left(\frac{b}{2} \right)^2 - 0 \right] = (m_1 + \frac{1}{4} m_2) b^2 \\
&= I_{33} \\
I_{12} &= -m_1 b \cdot 0 - m_2 \left(\frac{b}{2} \right) \cdot 0 = 0 \\
\text{likewise all } I_{i \neq j} &= 0
\end{aligned}
$$

Thus

$$
\{\mathbf{I}\} = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & (m_1 + \frac{1}{4} m_2) b^2 & 0 \\ 0 & 0 & (m_1 + \frac{1}{4} m_2) b^2 \end{array} \right\}
$$

The torque on the masses due to gravity is

$$
\begin{aligned}
\mathbf{N} &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha \\
&= \sum_\alpha \mathbf{r}_\alpha \times m_\alpha \mathbf{g}_\alpha \\
\text{where} \quad \mathbf{g} &= g \cos\theta \hat{\mathbf{x}} - g \sin\theta \hat{\mathbf{y}} \\
\text{and} \quad \mathbf{r}_\alpha &= r_\alpha \hat{\mathbf{x}} \\
\text{so} \quad \mathbf{r}_\alpha \times \mathbf{g} &= -r_\alpha g \sin\theta (\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}) \\
\text{and} \quad \mathbf{N} &= -m_1 b g \sin\theta \hat{\mathbf{z}} - m_2 \frac{b}{2} g \sin\theta \hat{\mathbf{z}} \\
&= -\left(m_1 + \frac{1}{2} m_2 \right) b g \sin\theta \hat{\mathbf{z}}
\end{aligned}
$$

Since $\mathbf{N} = \{\mathbf{I}\} \cdot \dot{\vec{\omega}}$,

$$\begin{pmatrix} 0 \\ 0 \\ (m_1 + \frac{1}{2}m_2)bg\sin\theta \end{pmatrix} = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & (m_1 + \frac{1}{4}m_2)b^2 & 0 \\ 0 & 0 & (m_1 + \frac{1}{4}m_2)b^2 \end{array} \right\} \cdot \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta} \end{pmatrix}$$

$$\begin{aligned} \Rightarrow (m_1 + \frac{1}{2}m_2)bg\sin\theta &= (m_1 + \frac{1}{4}m_2)b^2\ddot{\theta} \\ \text{or} \quad \ddot{\theta} &\simeq \omega_0^2\theta \\ \text{where} \quad \omega_0 &= \sqrt{\frac{m_1 + \frac{1}{2}m_2}{m_1 + \frac{1}{4}m_2}\left(\frac{g}{b}\right)} \end{aligned}$$

is the frequency of small oscillations.

Note that you could also have solved this problem using the Lagrange equations.

Since $\{\mathbf{I}\}$ is diagonal, our coordinate system is evidently the pendulum's principal axes. In this case, the system's Lagrangian L is

$$\begin{aligned} L &= T + U \\ \text{where} \quad T &= T_{rot} = \frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2 \\ \text{where} \quad I_i &= \text{ the diagonal elements} \\ \text{so} \quad T &= \frac{1}{2}I_{33}\omega^2 = \frac{1}{2}(m_1 + \frac{1}{4}m_2)b^2\dot{\theta}^2 \\ \text{and} \quad U &= -m_1 gb\cos\theta - m_2 g\frac{b}{2}\cos\theta \end{aligned}$$

The resulting Lagrange will lead to the same equation of motion, namely $\ddot{\theta} \simeq \omega_0^2\theta$.

Determining a body's Principal Axes

Evidently your equations of motion simplify considerably when you choose a coordinate system parallel to the body's principal axes of inertia.

How do you determine a body's principal axes?

Obtaining the solution can be laborious for an irregularly shaped body. However it is less painful for a symmetric body...

Start with a rigid body,
choose a coordinate system,
and calculate its inertia tensor $\{\mathbf{I}\}$.
At this point the I_{ij} are known quantities.

Now let the body rotate about one of the three principal axes.
If we knew the orientation of the principal axes, we could adopt those axes as our coordinate system.
In this *principal axes coordinate system*, the inertia tensor $\{\mathbf{I}\}$ is diagonal.
Let $I =$ any one of those three (unknown) elements on the diagonal,

$$\begin{aligned} \text{so} \quad \mathbf{L} &= I\vec{\omega} \\ \text{or} \quad L_i &= I\omega_i \end{aligned}$$

Since the body is assumed to be rotating about one of its principal axes, $\vec{\omega} = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$ indicates the orientation of that particular axis.
Thus we want to solve for $(\omega_1, \omega_2, \omega_3)$ which gives the orientation of that principal axis.

$$\begin{aligned} \text{since} \quad L_i &= \sum_j I_{ij}\omega_i = I\omega_i \\ \sum_j I_{ij}\omega_i &- I\omega_i = 0 \end{aligned}$$

$$
\begin{aligned}
\text{thus} \quad (I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \quad \text{for } i = 1 && (1) \\
I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 &= 0 \quad \text{for } i = 2 && (2) \\
I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 &= 0 \quad \text{for } i = 3 && (3)
\end{aligned}
$$

The only known quantities in the above are the I_{ij};
the I and the ω_i are still unknown.

Nonetheless, the above system of equations has a nontrivial solution for the ω_i when the determinant of the coefficients is zero:

$$
\begin{vmatrix}
(I_{11} - I) & I_{12} & I_{13} \\
I_{21} & (I_{22} - I) & I_{23} \\
I_{31} & I_{32} & (I_{33} - I)
\end{vmatrix} = 0
$$

This is the characteristic equation for the matrix $\{\mathbf{I}\}$,
which yields a cubic equation for I which has three roots I_1, I_2, and I_3,
which are the *principal moments of inertia.*

If you were to compute the inertia tensor in the principal axes coordinate system, these numbers would be the three nonzero elements along the diagonal.

Now for the laborious part of the solution...

Next, set $I = I_1$ in equations (1-3) and simultaneously solve these equations for $(\omega_1, \omega_2, \omega_3)$.

This yields the orientation of the first principal axis $\vec{\omega}_1 = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$.

Then set $I = I_2$ and again solve for the ω_i, which yields the next principal axis $\vec{\omega}_1$.

And likewise for $I = I_3$.

Example 11.5

Find the principal axes for the earlier cube problem with the origin at a corner. Recall that

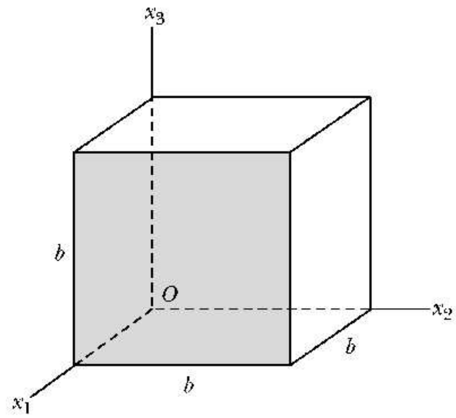

Fig. 11-3.

$$\{\mathbf{I}\} = Mb^2 \begin{Bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{Bmatrix}$$

which has a characteristic equation

$$Mb^2 \begin{vmatrix} (2/3 - I^*) & -1/4 & -1/4 \\ -1/4 & (2/3 - I^*) & -1/4 \\ -1/4 & -1/4 & (2/3 - I^*) \end{vmatrix} = 0$$

where $I^* \equiv I/Mb^2$.

Thus the principal moments of inertia are the roots I^* that satisfy

$$(2/3 - I^*)\left[(2/3 - I^*)^2 - \frac{1}{16}\right] + \frac{1}{4}\left[-\frac{1}{4}(2/3 - I^*) - \frac{1}{16}\right] - \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4}(2/3 - I^*)\right] = 0$$

which has roots $I_1^* = \frac{1}{6}$, $I_2^* = \frac{11}{12}$, and $I_3^* = \frac{11}{12}$ (see page 427).

Now set $I = I_1^* Mb^2 = \frac{1}{6}Mb^2$ in the equation $\mathbf{L} - I_1\vec{\omega}_1 = 0$
to obtain the orientation of the first principal axis $\vec{\omega}_1 = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$:

$$\begin{aligned} \left(\frac{2}{3} - \frac{1}{6} = \frac{1}{2}\right)\omega_1 - \frac{1}{4}\omega_2 - \frac{1}{4}\omega_3 &= 0 \\ -\frac{1}{4}\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{4}\omega_3 &= 0 \\ -\frac{1}{4}\omega_1 - \frac{1}{4}\omega_2 + \frac{1}{2}\omega_3 &= 0 \end{aligned}$$

with Mb^2 already factored out.

Inspection shows that this set of equations has solution $\omega_1 = \omega_2 = \omega_3$,
so the first principal axis points in the direction $\vec{\omega}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$.

This vector points from the origin at the cube's corner along the diagonal to the far corner.

Keep in mind that only the direction is important here, not the magnitude.

Next, repeat this procedure for the other roots $I_2 = \frac{11}{12}Mb^2 = I_3$
to determine the other two principal axes $\vec{\omega}_2$ and $\vec{\omega}_3$.

However if you actually do just that,
you will *not* get any useful constraints on the ω_i!
This means that the choice of other two axes are *arbitrary*, almost...

Remember that our derivation assumes that the principal axes are a right–handed Cartesian coordinate system, so the other two axes must be perpendicular to $\vec{\omega}_1$ as well as mutually orthogonal.

$$\begin{aligned} \text{Thus} \quad \vec{\omega}_2 &= \hat{\mathbf{x}} - \hat{\mathbf{y}} \\ \text{and} \quad \vec{\omega}_3 &= -\hat{\mathbf{x}} - \hat{\mathbf{y}} + 2\hat{\mathbf{z}} \end{aligned}$$

would be a valid choice of principal axes.

Euler's equations for a rigid body

The following will derive *Euler's equations* for a rigid body, which describe how a rigid body's orientation is altered when a torque is applied.

Recall that a rotating rigid body has angular momenta components

$$L_i = \sum_{j=1}^{3} I_{ij}\omega_j \qquad \text{(see Eq. 11.20a)}$$

$$\text{so} \quad \mathbf{L} = \sum_{i=1}^{3} L_i \hat{\mathbf{x}}_i = \sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\omega_j \hat{\mathbf{x}}_i = \text{ total angular momentum vector}$$

Keep in mind that the $\hat{\mathbf{x}}_i$ are the axes in the *rotating* coordinate system.

Now exert a external torque $\mathbf{N} = \dot{\mathbf{L}}$ on the body.
And as is usual, all time derivatives are to be computed in the *fixed* reference frame. Thus

$$\begin{aligned} \mathbf{N} &= \frac{d}{dt}\sum_{ij} I_{ij}\omega_j \hat{\mathbf{x}}_i \\ &= \sum_{ij} I_{ij}\left[\dot{\omega}_j \hat{\mathbf{x}}_i + \omega_j \left(\frac{d\hat{\mathbf{x}}_i}{dt}\right)_{fixed}\right] \end{aligned}$$

Recall that for any vector $\mathbf{Q}$,

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{fixed} = \left(\frac{d\mathbf{Q}}{dt}\right)_{rotating} + \vec{\omega} \times \mathbf{Q} \qquad \text{(Eqn. 10.12)}$$

$$\text{so} \quad \left(\frac{d\hat{\mathbf{x}}_i}{dt}\right)_{fixed} = \vec{\omega} \times \hat{\mathbf{x}}_i$$

Thus

$$\mathbf{N} = \sum_{ij} I_{ij}\left(\dot{\omega}_j \hat{\mathbf{x}}_i + \omega_j \vec{\omega} \times \hat{\mathbf{x}}_i\right)$$

Now lets choose our coordinate system to be the body's principal axes, so $I_{ij} = I_i\delta_{ij}$ and

$$
\begin{aligned}
\mathbf{N} &= \sum_i I_i \left(\dot{\omega}_i \hat{\mathbf{x}}_i + \omega_i \vec{\omega} \times \hat{\mathbf{x}}_i\right) \\
&= I_1\dot{\omega}_1\hat{\mathbf{x}} + I_2\dot{\omega}_2\hat{\mathbf{y}} + I_3\dot{\omega}_3\hat{\mathbf{z}} + I_1\omega_1\vec{\omega}\times\hat{\mathbf{x}} + I_2\omega_2\vec{\omega}\times\hat{\mathbf{y}} + I_3\omega_3\vec{\omega}\times\hat{\mathbf{z}}
\end{aligned}
$$

Do the cross products. First note that

$$
\begin{aligned}
\hat{\mathbf{x}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}} \\
\hat{\mathbf{y}} \times \hat{\mathbf{z}} &= \hat{\mathbf{x}} \\
\hat{\mathbf{z}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}} \\
\text{and} \quad \vec{\omega} &= \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}} = \text{ the body's spin axis} \\
\text{so} \quad \vec{\omega}\times\hat{\mathbf{x}} &= \omega_2\hat{\mathbf{y}}\times\hat{\mathbf{x}} + \omega_3\hat{\mathbf{z}}\times\hat{\mathbf{x}} = -\omega_2\hat{\mathbf{z}} + \omega_3\hat{\mathbf{y}} \\
\vec{\omega}\times\hat{\mathbf{y}} &= \omega_1\hat{\mathbf{z}} - \omega_3\hat{\mathbf{x}} \\
\vec{\omega}\times\hat{\mathbf{z}} &= -\omega_1\hat{\mathbf{y}} + \omega_2\hat{\mathbf{x}}
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\mathbf{N} &= \hat{\mathbf{x}}(I_1\dot{\omega}_1 - I_2\omega_2\omega_3 + I_3\omega_3\omega_2) + \hat{\mathbf{y}}(I_2\dot{\omega}_2 + I_1\omega_1\omega_3 - I_3\omega_3\omega_1) \\
&+ \hat{\mathbf{z}}(I_3\dot{\omega}_3 - I_1\omega_1\omega_2 + I_2\omega_2\omega_1)
\end{aligned}
$$

The three components of this equation,

$$
\begin{aligned}
N_x &= I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \\
N_x &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\
N_x &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2
\end{aligned}
$$

are known as *Euler's equations*, which relate the torque N_i to the time–rate of change $\dot{\omega}_i$ in the rigid body's orientation.

How to use Euler's eqns.

Suppose you want to know what the torque **N** does to a rigid body. In other words, how does this torque alter the body's spin axis $\vec{\omega}$ and re–orient its principal axes?

1. Choose the orientation of rotating coordinate system at some instant of time t_0.

2. Make note of $\vec{\omega}$ and **N** at time $t = t_0$

3. Calculate $\{\mathbf{I}\}$ and use the method described in Section 11.5 do determine the body's principal moments of inertia I_1, I_2, I_3 and the orientation of its principal axes $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ at time t_0

4. Use Euler's Equations to obtain $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$

5. Solve those equations (if possible) to obtain $\omega_1(t), \omega_2(t), \omega_3(t)$. This give you the time–history of the body's rotation axis $\vec{\omega}$.

6. Now solve for the body's changing orientation, ie, determine the time–history of the body's principal axes $\hat{\mathbf{x}}_i(t)$ which evolve according to

$$\frac{d\hat{\mathbf{x}}_i}{dt} = \vec{\omega}(t) \times \hat{\mathbf{x}}_i$$

solving this equation completely specifies the body's motion over time.

Good luck...

Example 11.10—A simple application of Euler's Eqns.

What torque must be exerted on the dumbbell in order to maintain this motion?

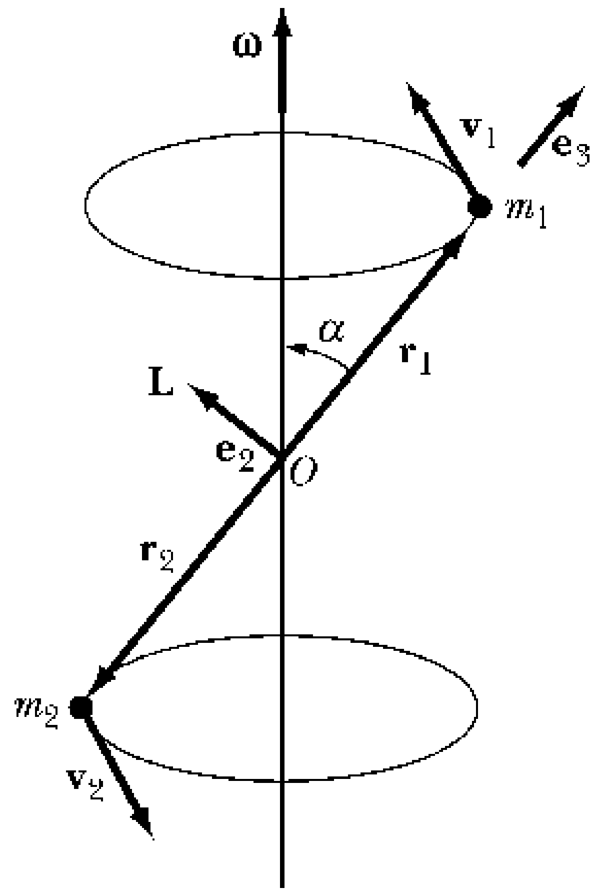

Fig. 11-11.

1. First choose the orientation of the rotating coordinate system:
set $\hat{\mathbf{z}}$ along the length of the rod
since $\hat{\mathbf{L}}$ is perpendicular to the rod, let this direction=$\hat{\mathbf{y}}$
and $\hat{\mathbf{x}}$ is chosen to complete this right–handed coordinate system.

2. From the diagram, $\vec{\omega} = \omega \sin\alpha \hat{\mathbf{y}} + \omega \cos\alpha \hat{\mathbf{z}}$
so $\omega_1 = 0$, $\omega_2 = \omega \sin\alpha$, and $\omega_3 = \omega \cos\alpha$

The torque $\mathbf{N} = N_x \hat{\mathbf{x}} + N_y \hat{\mathbf{y}} + N_z \hat{\mathbf{z}}$ is to be determined.

3. Calculate $\{\mathbf{I}\}$:

Use $I_{ij} = \sum_\alpha m_\alpha(\delta_{ij} \sum_k x^2_{\alpha,k} - x_{\alpha,i}x_{\alpha,j})$ to show that

$$\{\mathbf{I}\} = \left\{ \begin{array}{ccc} (m_1+m_2)b^2 & 0 & 0 \\ 0 & (m_1+m_2)b^2 & 0 \\ 0 & 0 & 0 \end{array} \right\}$$

evidently our coordinate system *is* this body's principal axes. Yea!
The principal moments of inertia are thus $I_1 = I_2 = (m_1+m_2)b^2$ and $I_3 = 0$.

4. Note that the $\dot{\omega}_i = 0$. Thus Euler's eqns. are:

$$\begin{aligned} N_x &= I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = -(m_1+m_2)b^2\omega^2 \sin\alpha\cos\alpha \\ N_y &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0 \\ N_z &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \end{aligned}$$

The torque that keeps the dumbbell rotating about the fixed axis $\vec{\omega}$ is $\mathbf{N} = -(m_1+m_2)b^2\omega^2 \sin\alpha\cos\alpha\hat{\mathbf{x}}$.

What forces $\mathbf{F}_1$ and $\mathbf{F}_2$ on masses m_1 and m_2 give rise to this torque $\mathbf{N}$?

Force–free motion of a symmetric top

The symmetric top has azimuthal symmetry about one axis, assumed here to be the $\hat{\mathbf{z}} = \hat{\mathbf{x}}_3$ axis.

A symmetric top also have $I_1 = I_2 \neq I_3$.

Did you encounter any symmetric tops in the homework assignment?

The 'force–free' means torque $\mathbf{N} = 0$.

For this problem, Euler's eqns. become:

$$\begin{aligned} I_1\dot{\omega}_1 - (I_1 - I_3)\omega_2\omega_3 &= 0 \\ I_1\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

where $\vec{\omega} = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$ is the body's spin–axis vector.

Keep in mind that the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes are the body's principal axes that corotate with the body.

Principal axis rotation corresponds to the case where $\vec{\omega}$ is parallel to any one of the body's principal axes, say, $\vec{\omega} = \omega_2\hat{\mathbf{y}}$.
How does that body's spin–axis $\vec{\omega}$ evolve over time?

Non–principal axis rotation is the more interesting case where $\vec{\omega} = \omega_1\hat{\mathbf{x}} + \omega_2\hat{\mathbf{y}} + \omega_3\hat{\mathbf{z}}$ where at least two of the ω_i are nonzero.

Note that $\dot{\omega}_3 = 0$ so $\omega_3 =$ constant.

Then

$$\begin{aligned}
\dot{\omega}_1 &= -\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 = -\Omega\omega_2 \\
\dot{\omega}_2 &= +\left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_1 = +\Omega\omega_1 \\
\text{where} \quad \Omega &\equiv \frac{I_3 - I_1}{I_1}\omega_3 = \text{ constant}
\end{aligned}$$

Solving these coupled equations will give you the time–history of the top's spin–axis $\vec{\omega}$.

To solve, set

$$\begin{aligned}
\eta(t) &\equiv \omega_1(t) + i\omega_2(t) \\
\text{so} \quad \dot{\eta} &= \dot{\omega}_1 + i\dot{\omega}_2 \\
&= -\Omega\omega_2 + i\Omega\omega_1 \\
\text{or} \quad \dot{\eta} &= i\Omega\eta \\
\text{so} \quad \eta(t) &= Ae^{i\Omega t} \\
&= A\cos\Omega t + iA\sin\Omega t \\
&= \omega_1 + i\omega_2 \\
\Rightarrow \quad \omega_1(t) &= A\cos\Omega t \\
\omega_2(t) &= A\sin\Omega t \\
\text{while} \quad \omega_3 &= \text{ constant}
\end{aligned}$$

Note that ω_1 and ω_2 are the projections of the $\vec{\omega}$ axis onto the $\hat{\mathbf{x}}$–$\hat{\mathbf{y}}$ plane, which traces a circle around the $\hat{\mathbf{z}}$ axis with angular velocity Ω.

Evidently the spin–axis *precesses* about the body's symmetry axis $\hat{\mathbf{z}}$ when $\vec{\omega}$ is *not* aligned with one of the body's principal axes.

An observer who co–rotates with the top will see this precessing spin–axis vector $\vec{\omega}$ sweep out a cone, called the *body cone*.

What does the observer in the fixed reference frame see?

First note that the system's total energy is pure rotational:

$$\begin{aligned} E &= T_{rot} = \frac{1}{2}\vec{\omega}\cdot\mathbf{L} = \frac{1}{2}\omega L\cos\phi \\ \text{where} \quad \phi &= \text{angle between } \vec{\omega} \text{ and angular momentum } \mathbf{L} \end{aligned}$$

Since E is constant for this conservative system,
and $\mathbf{L}$ is constant for this torque–free system,
$\omega\cos\phi =$ constant $\Rightarrow$ the projection of $\vec{\omega}$ onto $\mathbf{L}$ is constant.

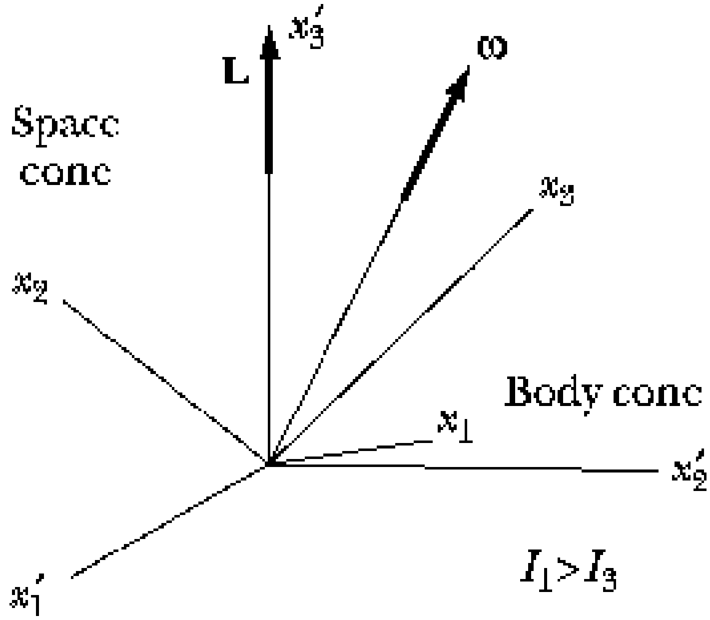

Fig. 11-13.

The observer in the fixed reference frame thus sees the spin–axis vector $\vec{\omega}$ precess about the angular momentum vector $\mathbf{L}$, which causes $\vec{\omega}$ to sweep out a *space cone.*

Note that the spin axis vector $\vec{\omega}$ always points at the spot where the two cones touch.

The three vectors $\mathbf{L}$, $\vec{\omega}$, and $\hat{\mathbf{x}}_3$ all inhabit the same plane (the proof given on page 450).

Thus the inertial observer sees both the spin axis $\vec{\omega}$ and the body's symmetry axis $\hat{\mathbf{x}}_3$ precess about the fixed angular momentum vector $\mathbf{L}$ at the same rate.